A new approach toward boundedness in a two-dimensional parabolic chemotaxis system with singular sensitivity

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We consider the parabolic chemotaxis model

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v) \\ v_t = \Delta v - v + u \end{cases}$$

in a smooth, bounded, convex two-dimensional domain and show global existence and boundedness of solutions for $\chi \in (0, \chi_0)$ for some $\chi_0 > 1$, thereby proving that the value $\chi = 1$ is not critical in this regard.
Our main tool is consideration of the energy functional

$$\mathcal{F}_{a,b}(u,v) = \int_\Omega u \ln u - a \int_\Omega u \ln v + b \int_\Omega |\nabla \sqrt{v}|^2$$

for $a > 0$, $b \geq 0$, where using nonzero values of b appears to be new in this context.
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1 Introduction

Numerous phenomena in connection with spontaneous aggregation can be described by PDE models incorporating a cross-diffusion mechanism. A prototypical example, which lies at the core of models used for a variety of purposes and to so different aims as the description pattern formation of bacteria or slime mold in biology [11] or the prediction of burglary in criminology [14], is the following variant of the Keller-Segel system of chemotaxis:

$$\begin{aligned} u_t =& \Delta u - \nabla \cdot (u\, S(v)\, \nabla v) \\ v_t =& \Delta v - v + u \\ \partial_\nu u|_{\partial\Omega} =& \partial_\nu v|_{\partial\Omega} = 0 \\ u(\cdot, 0) = u_0&, \, v(\cdot, 0) = v_0 \end{aligned} \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, with given nonnegative initial data u_0, v_0. We shall be concerned with the case of the singular sensitivity function S given by

$$S(v) = \frac{\chi}{v} \tag{2}$$

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for a constant $\chi > 0$, which is in compliance with the Weber-Fechner law of stimulus perception (see [12]).
One of the first questions of mathematical interest with respect to this model is that of existence of a global classical solution, as opposed to blow-up of solutions in finite time. For the vast mathematical literature on chemotaxis, a large part of which is concerned with this question, see one of the survey articles [9, 10, 7, 1] and references therein.
According to the standard reasoning in the realm of chemotaxis equations (as e.g. formulated in [1]), in order to obtain global existence of classical solutions, for the two-dimensional case considered here, it is sufficient to derive t-independent bounds on the quantities $\int_\Omega u(t) \ln u(t)$ and $\int |\nabla v(t)|^2$.
To achieve this in the particular context of (1), it has proven useful to consider the expression

$$\int_\Omega u \ln u - a \int_\Omega u \ln v, \tag{3}$$

as it has been done by Nagai, Senba, Yoshida [15] or Biler [2]. In these works, global existence of solutions has been derived for $\chi \leq 1$.
In the present article we shall answer the question whether $\chi = 1$ is a critical value in this regard in the negative. This question had been left open in [20], where the above-mentioned results have been generalized to higher dimension n, then obtaining existence in the case $\chi < \sqrt{2/n}$.
Let us mention some more results concerning equation (1): That the classical solutions for $\chi < \sqrt{2/n}$ are global-in-time bounded has been shown in [4]. In [20] also weak solutions have been shown to exist for (1), as long as $\chi < \sqrt{\frac{n+2}{3n-4}}$. In the radially symmetric setting, moreover, certain global weak "power-λ-solutions" exist ([17]).
Related parabolic-elliptic chemotaxis models are investigated, e.g. in [5], where the presence of terms describing logistic growth is used to ensure global existence and boundedness of classical solutions. In [6] global existence and boundedness of classical solutions to the parabolic-elliptic counterpart of (1) are obtained for even more singular sensitivities of the form $0 < S(v) \leq \frac{\chi}{v^k}$, $k \geq 1$, under a smallness condition on χ, which for $k = 1$ and $n = 2$ amounts to $\chi < 1$.
Also concerning classical solutions of the fully parabolic system (1), to the best of our knowledge, the assertions for $\chi \leq 1$ are the best known so far.
Since the new possible values for χ are but slightly larger than 1, rather than these values it is the method that can be considered the new contribution of the present article: Key to our approach toward the expansion of the interval of values for χ known to yield global solutions, namely, shall be the employment of an additional summand

$$b \int |\nabla \sqrt{v}|^2$$

in (3). Functionals containing this term have successfully been used in the context of coupled chemotaxis-fluid systems (see [21]) or of chemotaxis models with consumption of the chemoattractant [18] (e.g. obtained from the aforementioned system upon neglection of the fluid).
In the end we will arrive at the following

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a convex bounded domain with smooth boundary. Let $0 \leq u_0 \in C^0(\overline{\Omega})$, $u_0 \not\equiv 0$, $0 < v_0 \in \bigcup_{q>2} W^{1,q}(\Omega)$. Then there exists $\chi_0 > 1$ such that for any $\chi \in (0, \chi_0)$ the system (1) has a global classical solution, which is bounded.*

The plan of the paper is as follows: In the next section we will discuss local existence of and an extensibility criterion for solutions to (1). Section 3 provides identities and estimates that will facilitate the usage of the additional term at the center of the proof of Theorem 1.1, to which Section 4 will be devoted.

2 How to ensure global existence

A general existence theorem for chemotaxis models is the following, taken from [1]:

Theorem 2.1. *Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and let $q > n$. For some $\omega \in (0,1)$ let $S \in C^{1+\omega}_{loc}(\overline{\Omega} \times [0,\infty) \times \mathbb{R}^2)$, $f \in C^{1-}(\overline{\Omega} \times [0,\infty) \times \mathbb{R}^2)$ and $g \in C^{1-}_{loc}(\overline{\Omega} \times [0,\infty) \times \mathbb{R}^2)$, and assume that $f(x,t,0,v) \geq 0$ for all $(x,t,v) \in \overline{\Omega} \times [0,\infty)^2$ and that $g(x,t,u,0) \geq 0$ for any $(x,t,u) \in \overline{\Omega} \times [0,\infty)^2$. Then for all nonnegative $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,q}(\Omega)$ there exist $T_{max} \in (0,\infty]$ and a uniquely determined pair of nonnegative functions*

$$\begin{aligned} u \in& C^0(\overline{\Omega} \times [0,T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max})), \\ v \in& C^0(\overline{\Omega} \times [0,T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max})) \cap L^\infty_{loc}([0,T_{max}); W^{1,q}(\Omega)), \end{aligned} \tag{4}$$

such that (u,v) solves

$$\begin{aligned} u_t =& \Delta u - \nabla \cdot (uS(x,t,u,v)\nabla v) + f(x,t,u,v), \\ v_t =& \Delta v - v + g(x,t,u,v), \\ 0 =& \partial_\nu u|_{\partial\Omega} = \partial_\nu v|_{\partial\Omega}, \\ &u(\cdot,0) = u_0, v(\cdot,0) = v_0 \end{aligned} \tag{5}$$

classically in $\Omega \times (0,T_{max})$ and such that

$$\text{if } T_{max} < \infty, \text{ then } \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,q}(\Omega)} \to \infty \text{ as } t \nearrow T_{max}. \tag{6}$$

Proof. A Banach-type fixed point argument provides existence of mild solutions on a short time interval whose length T depends on $\|u_0\|_\infty$, $\|v_0\|_{W^{1,q}}$. Standard bootstrapping arguments ensure the regularity properties listed above. It follows from the dependence of T on the norms of u_0 and v_0 that the solution can be extended to $T_{max} \in (0,\infty]$ satisfying (6), see [1, Lemma 4.1]. ◻

This theorem is not directly applicable to (1), because it does not cover the case of singular functions S. We will remove this obstruction via use of the following lemma, which is a generalization of Lemma 2.2 of [4].

Lemma 2.2. *Let the conditions of Theorem 2.1 be satisfied and let $\zeta > 0$.*
Then there is $\eta = \eta(u_0, v_0, \zeta) > 0$ such that if v_0 and the solution (u,v) to (5) satisfy

$$\inf v_0 > 0 \qquad \text{and} \qquad \inf_{s \in [0,T_{max})} \int_\Omega g(x,s,u(x,s),v(x,s))dx \geq \zeta,$$

the second component of the solution also fulfils

$$v(x,t) \geq \eta \qquad \text{for all } (x,t) \in \overline{\Omega} \times [0,T_{max}).$$

Proof. Let us fix $\tau = \tau(u_0,v_0) > 0$ such that

$$\inf_\Omega v(\cdot,t) \geq \frac{1}{2} \inf_\Omega v_0 \qquad \text{for all } t \in [0,\tau].$$

Employing the pointwise estimate

$$(e^{t\Delta}w)(x) \geq \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{d^2}{4t}} \int_\Omega w \qquad \text{for nonnegative } w \in C^0(\overline{\Omega})$$

for the Neumann heat semigroup $e^{t\Delta}$ with $d = \operatorname{diam}\Omega$, as provided in [4, Lemma 2.2] following [8, Lemma 3.1], we can then conclude that

$$\begin{aligned} v(\cdot,t) =& e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)} g(\cdot,s,u(\cdot,s),v(\cdot,s))ds \\ \geq& \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} e^{-\frac{d^2}{4(t-s)}-(t-s)} \int_\Omega g(\cdot,s,u(\cdot,s),v(\cdot,s))ds \\ \geq& \int_0^t \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{-(r+\frac{d^2}{4r})} dr \inf_{s\in[0,t]} \int_\Omega g(x,s,u(x,s),v(x,s))dx \end{aligned}$$

$$\geq \zeta \int_0^{\tau} \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{-(r+\frac{d^2}{4r})} dr \qquad \text{in } \Omega$$

for any $t \in [\tau, T_{max})$. With $\eta = \min\{\frac{\inf_\Omega v_0}{2}, \zeta \int_0^{\tau(u_0,v_0)} \frac{1}{(4\pi r)^{\frac{n}{2}}} e^{-(r+\frac{d^2}{4r})} dr\}$ this proves the claim. ☐

With this lemma we can weaken the assumptions on the sensitivity S so as to allow for a singularity at $v = 0$.

Theorem 2.3. *i) Let $S \in C^{1+\omega}_{loc}(\overline{\Omega} \times [0,\infty) \times \mathbb{R} \times (0,\infty))$ for some $\omega \in (0,1)$ and apart from the condition on S let the assumptions of Theorem 2.1 be satisfied.*
Additionally, assume that f is nonnegative and $g(x,t,u,v) \geq cu$ for some $c > 0$ and any $(x,t,u,v) \in \overline{\Omega} \times [0,\infty) \times \mathbb{R}^2$ and that $\inf_\Omega v_0 > 0$ and $\int_\Omega u_0 =: m > 0$. Then there is $T_{max} > 0$ such that (1) has a unique solution (u,v) as in (4) and such that (6) holds.
ii) Furthermore, if there are $K_1, K_2 > 0$ such that $f(x,t,u,v) \leq K_1$ and $g(x,t,u,v) \leq K_2(1+u)$ for all $(x,t,u,v) \in \Omega \times (0,\infty)^3$, and for every $\eta > 0$, $|S|$ is bounded on $\Omega \times (0,\infty)^2 \times (\eta,\infty)$, and if $n = 2$ and there is $M > 0$ such that

$$\int_\Omega u(\cdot,t) \ln u(\cdot,t) \leq M, \qquad \textit{and} \qquad \int_\Omega |\nabla v(\cdot,t)|^2 \leq M \qquad \textit{for all } t \in [0,T_{max}) \tag{7}$$

then (u,v) is global and bounded.

Proof. i) Let $\eta := \eta(u_0, v_0, cm)$ be as in Lemma 2.2. Let $\zeta \colon \mathbb{R} \to [0,1]$ be a smooth, monotone decreasing function with $\zeta(\frac{\eta}{2}) = 1$ and $\zeta(\eta) = 0$. Define

$$S_\eta(x,t,u,v) := \begin{cases} S(x,t,u,\frac{\eta}{2}), & (x,t,u,v) \in \overline{\Omega} \times [0,\infty) \times \mathbb{R} \times (-\infty,\frac{\eta}{2}), \\ \zeta(v)S(x,t,u,\frac{\eta}{2}) + (1-\zeta(v))S(x,t,u,v), & (x,t,u,v) \in \overline{\Omega} \times [0,\infty) \times \mathbb{R} \times [\frac{\eta}{2},\infty). \end{cases}$$

Then $S_\eta \in C^{1+\omega}_{loc}(\overline{\Omega} \times [0,\infty) \times \mathbb{R}^2)$ and S and S_η agree for $v \geq \eta$. Let us denote by $(5)_\eta$ problem (5) with S replaced by S_η. Then we can apply Theorem 2.1 to $(5)_\eta$ and obtain a solution (u,v) with the required properties (4) and (6). Nonnegativity of f and integration of the first equation of $(5)_\eta$ entail that $\int_\Omega u(t) \geq m$ for all $t \in [0,T_{max})$ and accordingly $\int_\Omega g(x,t,u(x,t),v(x,t))dx \geq cm > 0$ for all $t \in [0,T_{max})$. Therefore, by Lemma 2.2, $v \geq \eta$ and hence (u,v) solves (5) as well.
In order to carry over the uniqueness statement from Theorem 5, we ensure that any solution of (5) also solves $(5)_\eta$ in $\Omega \times [0,T_{max})$: Let v be a solution of (5). Let $\varepsilon \in (0,\frac{\eta}{2})$ and define $t_0 = \inf\{t : \inf_\Omega v(t) < \varepsilon\} \in (0,\infty]$. Then (u,v) solves $(5)_\eta$ in $(0,t_0)$. Assume $t_0 < \infty$. Then by Lemma 2.2 and continuity of v, $v(x,t_0) \geq \eta > \varepsilon = \inf_\Omega v(\cdot,t)$ for all $x \in \Omega$, a contradiction.
ii) Since (u,v) is a solution of $(5)_\eta$, we can apply [1, Lemma 4.3], which turns (7) into a uniform-in-time bound on $\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,q}(\Omega)}$, thus asserting global existence by means of (6) and boundedness. ☐

Remark 2.4. *Throughout the remaining part of the article, we will assume that $\Omega \subset \mathbb{R}^2$ is a bounded, smooth domain, that $0 \leq u_0 \in C^0(\overline{\Omega})$, $q > 2$ and $v_0 \in W^{1,q}(\Omega)$, $\inf_\Omega v_0 > 0$ and $\int_\Omega u_0 =: m > 0$.*
Then, in particular, Theorem 2.3 is applicable to (1). Furthermore, any solution (u,v) of (1) satisfies

$$\int_\Omega u(t) = m \qquad \textit{for all } t \in [0,T_{max}). \tag{8}$$

For the purpose of using it in the next proof, let us recall the well-known Gagliardo-Nirenberg inequality:

Lemma 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Let $j \geq 0, k \geq 0$ be integers and $p,q,r,s > 1$. There are constants $c_1, c_2 > 0$ such that for any function $w \in L^q(\Omega) \cap L^s(\Omega)$ with $D^k w \in L^r(\Omega)$,*

$$\left\|D^j w\right\|_p \leq c_1 \left\|D^k w\right\|_r^\alpha \|w\|_q^{1-\alpha} + c_2 \|w\|_s,$$

whenever $\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{k}{n}\right)\alpha + \frac{1-\alpha}{q}$ and $\frac{j}{k} \leq \alpha < 1$.

Proof. Cf. [16, p. 126] ☐

Lemma 2.6. *Let (u,v) be a solution to (1), let $\tau=\min\{1,\frac{T_{max}}{2}\}$, and assume there exists $C>0$ such that*

$$\int_t^{t+\tau}\int_\Omega \frac{|\nabla u|^2}{u} \le C \quad \textit{for any } t\in(0,T_{max}-\tau)$$

and that

$$\int_\Omega u(t)\ln u(t) \le C \quad \textit{for any } t\in(0,T_{max})$$

Then $T_{max}=\infty$ and (u,v) is bounded.

Proof. Let $c_1,c_2>0$ be the constants yielded by Lemma 2.5 for $j=0$, $k=1$, $q=2$, $r=2$, $\alpha=\frac{1}{2}$. Then with m from (8),

$$\begin{aligned}\int_t^{t+\tau}\int_\Omega u^2 &= \int_t^{t+\tau} \left\|\sqrt{u}\right\|_{L^4(\Omega)}^4 \le \int_t^{t+\tau}\left(c_1\left\|\nabla\sqrt{u}\right\|_{L^2(\Omega)}^\alpha \left\|\sqrt{u}\right\|_{L^2(\Omega)}^{1-\alpha} + c_2\left\|\sqrt{u}\right\|_{L^2(\Omega)}\right)^4\\ &\le \int_t^{t+\tau}\left(c_1\left(\int_\Omega\frac{|\nabla u|^2}{4u}\right)^{\frac14} m^{\frac14}+c_2m^{\frac12}\right)^4 \le \frac{c_1^4mC}{4}+c_2^4m^2\end{aligned}$$

holds for any $t\in(0,T_{max}-\tau)$.
Multiplying the second equation of (1) by $-\Delta v$ and integrating, from Young's inequality we obtain

$$\int_\Omega|\nabla v(t)|^2 \le \int_\Omega|\nabla v_0|^2 - \int_0^t\int_\Omega|\Delta v|^2 - \int_0^t\int_\Omega|\nabla v|^2 + \frac12\int_0^t\int_\Omega u^2+\frac12\int_0^t\int_\Omega|\Delta v|^2 \qquad \text{on } (0,T_{max}),$$

that is, $y(t):=\int_\Omega|\nabla v(t)|^2$ satisfies the differential inequality $y'+y\le f$ on $[0,T_{max})$, where $f=\frac12\int_\Omega u^2$ satisfies $\int_t^{t+\tau}|f(s)|ds\le C$ for all $t\in(0,T_{max}-\tau)$ and with some constant $C>0$. Let z be a solution to $z'+z=f$, $z(0)=z_0=\int_\Omega|\nabla v_0|^2$ and observe that the variation-of-constants formula entails

$$\begin{aligned}z(t)-e^{-t}z_0 &= \int_0^t e^{-s}f(t-s)ds \le \sum_{k=0}^{\lfloor t/\tau\rfloor-1}\int_{k\tau}^{(k+1)\tau} e^{-s}|f(t-s)|ds + \int_{\tau\lfloor t/\tau\rfloor}^t e^{-s}|f(t-s)|ds\\ &\le \sum_{k=0}^{\lfloor t/\tau\rfloor-1} e^{-k\tau}C + C \le C\left(1+\frac{1}{1-e^{-\tau}}\right) \qquad \text{for } t\in(0,T_{max}),\end{aligned}$$

so that an ODE comparison yields boundedness of $y=\int_\Omega|\nabla v(t)|^2$.
Together with the second assumption, the bound on $\int_\Omega u\ln u$, this is sufficient to conclude global existence and boundedness of solutions by Theorem 2.3 ii). □

3 Some useful general estimates and identities

Lemma 3.1. *Let Ω be convex and let $w\in C^2(\overline{\Omega})$ satisfy $\partial_\nu w|_{\partial\Omega}=0$. Then for all $x\in\partial\Omega$ also $\partial_\nu|\nabla w(x)|^2\le 0$.*

Proof. This is Lemme 2.I.1 of [13]. □

Lemma 3.2. *For all positive $w\in C^2(\overline{\Omega})$ satisfying $\partial_\nu w|_{\partial\Omega}=0$*

$$\int_\Omega w|D^2\ln w|^2 = \int_\Omega\frac1w|D^2w|^2+\int_\Omega\frac{1}{w^2}|\nabla w|^2\Delta w-\int_\Omega\frac{|\nabla w|^4}{w^3}$$

Proof. This proof is also contained in the proof of [21, Lemma 3.2]. The equality rests on the pointwise identity

$$w|D^2\ln w|^2 = w\left(-\frac{1}{w^2}|\nabla w|^2+\frac1wD^2w\right)^2 = \frac{|\nabla w|^4}{w^3}+\frac1w|D^2w|^2-\frac{1}{w^2}\nabla|\nabla w|^2\cdot\nabla w$$

and integration by parts in the last term giving

$$-\int_\Omega \frac{1}{w^2}\nabla|\nabla w|^2\cdot\nabla w = \int_\Omega \frac{1}{w^2}|\nabla w|^2\Delta w - 2\int_\Omega \frac{|\nabla w|^4}{w^3}.$$

□

Lemma 3.3. *Let $w\in C^2(\overline{\Omega})$ be positive and satisfy $\partial_\nu w|_{\partial\Omega}=0$. Then*

$$-\int_\Omega \frac{1}{w}|\Delta w|^2 = -\int_\Omega \frac{1}{w}|D^2 w|^2 - \frac{3}{2}\int_\Omega \frac{|\nabla w|^2\Delta w}{w^2} + \frac{1}{2}\int_\Omega \frac{|\nabla w|^4}{w^3} + \frac{1}{2}\int_{\partial\Omega}\frac{1}{w}\partial_\nu|\nabla w|^2.$$

Proof. This results from [21, Lemma 3.1] upon the choice of $h(w)=\frac{1}{w}$. The proof can be found in [3, Lemma 2.3]. □

Lemma 3.4. *(i) For all positive $w\in C^2(\overline{\Omega})$ satisfying $\partial_\nu w|_{\partial\Omega}=0$,*

$$-\int_\Omega \frac{1}{w}|\Delta w|^2 = -\int_\Omega w|D^2\ln w|^2 - \frac{1}{2}\int_\Omega \frac{1}{w^2}|\nabla w|^2\Delta w + \frac{1}{2}\int_{\partial\Omega}\frac{1}{w}\partial_\nu|\nabla w|^2.$$

(ii) If furthermore Ω is convex, then

$$-\int_\Omega \frac{1}{w}|\Delta w|^2 \le -\int_\Omega w|D^2\ln w|^2 - \frac{1}{2}\int_\Omega \frac{1}{w^2}|\nabla w|^2\Delta w.$$

Proof. This is a direct consequence of the previous three lemmata. □

Lemma 3.5. *There is $c_0>0$ such that for all positive $w\in C^2(\overline{\Omega})$ fulfilling $\partial_\nu w|_{\partial\Omega}=0$ the following estimate holds:*

$$\int_\Omega w|D^2\ln w|^2 \ge c_0\int_\Omega \frac{|\nabla w|^4}{w^3}.$$

Proof. An even more general version of this lemma and its proof can be found in [21, Lemma 3.3]. □

Remark 3.6. *As can be seen from the referenced lemma, the constant in the above statement can be chosen to be $\frac{1}{(2+\sqrt{2})^2}$.*

4 The energy functional. Proof of Theorem 1.1

In this section let us investigate the energy functional defined by

$$\mathcal{F}_{a,b}(u(t),v(t)) = \int_\Omega u(t)\ln u(t) - a\int_\Omega u(t)\ln v(t) + b\int_\Omega |\nabla\sqrt{v(t)}|^2, \qquad t\in[0,T_{max}), \tag{9}$$

for nonnegative parameters a,b.
If we want to gain useful information from this functional, the upper bounds on its derivative that we will derive, should be accompanied by bounds for $\mathcal{F}_{a,b}$ from below. In order to ensure those, let us first provide the following estimate for solutions of (1).

Lemma 4.1. *Let (u,v) be a solution to (1). For any $p>0$ there is $C_p>0$ such that*

$$\int_\Omega v^p(t)\le C_p \qquad \textit{for any } t\in[0,T_{max}).$$

Proof. Since $t\mapsto \|u(t)\|_{L^1(\Omega)}$ is constant by (8), for $p\ge 1$ this is a consequence of Duhamel's formula for the solution of the second equation of (1) and estimates for the Neumann heat semigroup, which can e.g. be found in [19, Lemma 1.3]: They provide $C>0$ such that for all $t\in(0,T_{max})$,

$$\begin{aligned}\|v(t)\|_{L^p(\Omega)} &\le \left\|e^{t(\Delta-1)}v_0\right\|_{L^p(\Omega)} + \int_0^t \left\|e^{(t-s)(\Delta-1)}(u(s)-m)\right\|_{L^p(\Omega)} + \left\|e^{-(t-s)}m\right\|_{L^p(\Omega)} ds\\ &\le \|v_0\|_{L^p(\Omega)} + \int_0^t \left(C(1+(t-s)^{-\frac{n}{2}(1-\frac{1}{p})})e^{-(t-s)}\,\|u(s)-m\|_{L^1(\Omega)} + e^{-(t-s)}m|\Omega|^{\frac{1}{p}}\right) ds.\end{aligned}$$

The case $p\in(0,1)$ then follows from $v^p\le 1+v$. □

The following lemma gives bounds from below as well as means to turn boundedness of $\mathcal{F}_{a,b}(u,v)$ into boundedness of $\int_\Omega u \ln u$.

Lemma 4.2. *Let $a, b \geq 0$. For any solution (u,v) of (1), there is $\gamma \in \mathbb{R}$ such that*

$$\mathcal{F}_{a,b}(u,v) \geq \frac{1}{2}\int_\Omega u \ln u - \gamma \qquad \text{on } (0, T_{max}).$$

Proof. Denoting $m = \int_\Omega u(t)$ as in (8), we have

$$\mathcal{F}_{a,b}(u,v) \geq \frac{1}{2}\int_\Omega u \ln u + \int_\Omega u \ln \frac{u^{\frac{1}{2}}}{v^a} = \frac{1}{2}\int_\Omega u \ln u + m \int_\Omega \left(-\ln \frac{v^a}{u^{\frac{1}{2}}}\right) \frac{u}{m},$$

similar as in the proof of [2, Thm. 3]. Hence, following an idea from the proof of [15, Lemma 3.3] in applying Jensen's inequality with the probability measure $\frac{u}{m} d\lambda$ and the convex function $-\ln$, we obtain

$$\begin{aligned}
\mathcal{F}_{a,b}(u,v) \geq & \frac{1}{2}\int_\Omega u \ln u - m \ln \int_\Omega \frac{v^a u^{\frac{1}{2}}}{m} \\
\geq & \frac{1}{2}\int_\Omega u \ln u - m \ln \left(\frac{1}{m}\left(\int_\Omega v^{2a} \int_\Omega u\right)^{\frac{1}{2}}\right) \\
\geq & \frac{1}{2}\int_\Omega u \ln u + \frac{m}{2} \ln m - \frac{m}{2} \ln C_{2a}
\end{aligned}$$

after applying Hölder's inequality and with C_{2a} as in 4.1. ∎

Lemma 4.3. *Let $a, b \geq 0$. For any solution (u,v) of (1),*
i) $\mathcal{F}_{a,b}(u,v)$ is bounded below.
ii) If $\sup_{t\in[0,T_{max})} \mathcal{F}_{a,b}(u(t),v(t)) < \infty$ then $\sup_{t\in[0,T_{max})} \int_\Omega u(t) \ln u(t) < \infty$.

Proof. Both statements are immediate consequences of Lemma 4.2. ∎

Lemma 4.1 as well enables us to control the first two summands of $\mathcal{F}_{a,b}(u,v)$ from above by $\int_\Omega \frac{u^2}{v}$.

Lemma 4.4. *Let (u,v) be a solution to (1) and let $a > 0$. Then for any $\delta > 0$ there is $c_\delta > 0$ such that*

$$\int_\Omega u \ln u - a \int_\Omega u \ln v \leq \delta \int_\Omega \frac{u^2}{v} + c_\delta \qquad \text{on } [0, T_{max}).$$

Proof. Given $a > 0$ let $\varepsilon \in (0,1)$ be so small that $\frac{1+\varepsilon-2a\varepsilon}{1-\varepsilon} > 0$. There is $C_\varepsilon > 0$ such that for any $x > 0$ we have $\ln x \leq C_\varepsilon x^\varepsilon$. Therefore for any $\delta > 0$ Young's inequality and Lemma 4.1 provide $C_\delta > 0$ and $c_\delta > 0$ satisfying

$$\begin{aligned}
\int_\Omega u \ln u - a \int_\Omega u \ln v = \int_\Omega u \ln \frac{u}{v^a} \leq C_\varepsilon \int_\Omega \frac{u^{1+\varepsilon}}{v^{a\varepsilon}} \leq \delta \int_\Omega (u^{1+\varepsilon} v^{-\frac{1+\varepsilon}{2}})^{\frac{2}{1+\varepsilon}} + C_\delta \int_\Omega (v^{\frac{1+\varepsilon-2a\varepsilon}{2}})^{\frac{2}{1-\varepsilon}} \\
\leq \delta \int_\Omega \frac{u^2}{v} + c_\delta.
\end{aligned}$$
∎

With these preparations, we turn to the time derivative of $\mathcal{F}_{a,b}(u,v)$, beginning with the already investigated first part:

Lemma 4.5. *For any $a \geq 0$ and any solution (u,v) of (1),*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{a,0}(u,v)(t) = -\int_\Omega \frac{|\nabla u|^2}{u} + (\chi + 2a)\int_\Omega \frac{\nabla u \cdot \nabla v}{v} - a(\chi+1)\int_\Omega \frac{u|\nabla v|^2}{v^2} + a\int_\Omega u - a\int_\Omega \frac{u^2}{v}$$

holds on $(0, T_{max})$.

Proof. Using the first equation of (1) in $\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u\ln u - a\int_\Omega u\ln v\right)$ and integrating by parts we obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega u\ln u - a\int_\Omega u\ln v\right) &= \int_\Omega u_t\ln u + \int_\Omega u_t - a\int_\Omega u_t\ln v - a\int_\Omega \frac{u}{v}v_t\\
&= -\int_\Omega \frac{\nabla u}{u}\left(\nabla u - \chi\frac{u}{v}\nabla v\right) + a\int_\Omega \frac{\nabla v}{v}\left(\nabla u - \chi\frac{u}{v}\nabla v\right) - a\int_\Omega \frac{u}{v}(\Delta v - v + u)\\
&= -\int_\Omega \frac{|\nabla u|^2}{u} + \chi\int_\Omega \frac{\nabla u\cdot\nabla v}{v} + a\int_\Omega \frac{\nabla u\cdot\nabla v}{v} - a\chi\int_\Omega \frac{u|\nabla v|^2}{v^2}\\
&\quad + a\int_\Omega \frac{\nabla u\cdot\nabla v}{v} - a\int_\Omega \frac{u|\nabla v|^2}{v^2} + a\int_\Omega u - a\int_\Omega \frac{u^2}{v}.\end{aligned}$$ □

Since we do not know the sign of $\int_\Omega \frac{\nabla u\cdot\nabla v}{v}$ and, in this situation, cannot control $\int_\Omega \frac{u|\nabla v|^2}{v^2}$, we are left with Young's inequality, hoping that the resulting coefficient $\frac{(\chi+2a)^2}{4} - a(\chi+1)$ of $\int_\Omega \frac{u|\nabla v|^2}{v^2}$ turns out to be negative. This can be achieved if $\chi < 1$.
However, it becomes possible to cope with larger parameters if $\int_\Omega \frac{u|\nabla v|^2}{v^2}$ can be controlled, e.g. by having control over $\int_\Omega \frac{|\nabla v|^4}{v^3}$ and $\int_\Omega \frac{u^2}{v}$. The second term already being in place, fortunately, the first is one of the terms arising from the following:

Lemma 4.6. *Let Ω be convex. For any solution (u,v) of (1),*

$$4\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega |\nabla\sqrt{v}|^2\right) \le -2c_0\int_\Omega \frac{|\nabla v|^4}{v^3} - \int_\Omega \frac{|\nabla v|^2}{v} + 2\int_\Omega \frac{\nabla u\cdot\nabla v}{v} - \int_\Omega \frac{|\nabla v|^2 u}{v^2} \tag{10}$$

holds on $(0,T_{max})$, where c_0 is the constant provided by Lemma 3.5.

Proof. From the second equation of (1), we obtain

$$\begin{aligned}4\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega |\nabla\sqrt{v}|^2\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{|\nabla v|^2}{v} = \int_\Omega \frac{2\nabla v\cdot\nabla v_t}{v} - \int_\Omega \frac{|\nabla v|^2 v_t}{v^2}\\
&= \int_\Omega \frac{2\nabla v\cdot\nabla\Delta v}{v} - \int_\Omega \frac{2|\nabla v|^2}{v} + \int_\Omega \frac{2\nabla v\cdot\nabla u}{v} - \int_\Omega \frac{|\nabla v|^2\Delta v}{v^2} + \int_\Omega \frac{|\nabla v|^2 v}{v^2} - \int_\Omega \frac{|\nabla v|^2 u}{v^2}.\end{aligned}$$

Integration by parts in the first integral and merging the second and second to last summand lead us to

$$4\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega |\nabla\sqrt{v}|^2\right) = -2\int_\Omega \frac{|\Delta v|^2}{v} + \int_\Omega \frac{|\nabla v|^2\Delta v}{v^2} - \int_\Omega \frac{|\nabla v|^2}{v} + 2\int_\Omega \frac{\nabla u\cdot\nabla v}{v} - \int_\Omega \frac{|\nabla v|^2 u}{v^2}. \tag{11}$$

By Lemma 3.4 and due to the convexity of Ω we can transform the first summand according to

$$-2\int_\Omega \frac{|\Delta v|^2}{v} \le -2\int_\Omega v|D^2\ln v|^2 - \int_\Omega \frac{1}{v^2}|\nabla v|^2\Delta v,$$

making the second term in the right hand side of (11) vanish:

$$4\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega |\nabla\sqrt{v}|^2\right) \le -2\int_\Omega v|D^2\ln v|^2 + 2\int_\Omega \frac{\nabla u\cdot\nabla v}{v} - \int_\Omega \frac{|\nabla v|^2 u}{v^2}. \tag{12}$$

We are left with a term we can estimate with the help of Lemma 3.5:

$$-2\int_\Omega v|D^2\ln v|^2 \le -2c_0\int_\Omega \frac{|\nabla v|^4}{v^3},$$

thereby gaining the term which will make the crucial difference in the estimates to come and arriving at

$$4\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega |\nabla\sqrt{v}|^2\right) \le -2c_0\int_\Omega \frac{|\nabla v|^4}{v^3} - \int_\Omega \frac{|\nabla v|^2}{v} + 2\int_\Omega \frac{\nabla u\cdot\nabla v}{v} - \int_\Omega \frac{|\nabla v|^2 u}{v^2}.$$ □

If we combine the previous two lemmata, we are led to:

Lemma 4.7. *Let $\Omega \subset \mathbb{R}^2$ be a convex, bounded, smooth domain and let $a, b \geq 0$, $\delta \in (0,1)$. Then for any solution (u,v) of* (1),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{a,b}(u,v)(t) \leq \left(\frac{1}{4a(1-\delta)} \left(\frac{(\chi + 2a + \frac{b}{2})^2}{4(1-\delta)} - a\chi - a - \frac{b}{4} \right)_+^2 - \frac{bc_0}{2} \right) \int_\Omega \frac{|\nabla v|^4}{v^3}$$
$$- \delta \int_\Omega \frac{|\nabla u|^2}{u} - \delta \int_\Omega \frac{u^2}{v} + a \int_\Omega u - \frac{b}{4} \int_\Omega \frac{|\nabla v|^2}{v} \qquad \text{on } (0, T_{max}). \tag{13}$$

Proof. An estimate for $\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{a,b}(u(t),v(t))$ is given by the sum of the terms from Lemma 4.5 and Lemma 4.6:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{a,b}(u,v)(t) \leq - \int_\Omega \frac{|\nabla u|^2}{u} + \left(\chi + 2a + \frac{b}{2}\right) \int_\Omega \frac{\nabla u \cdot \nabla v}{v} - \left(a\chi + a + \frac{b}{4}\right) \int_\Omega \frac{u|\nabla v|^2}{v^2}$$
$$+ a \int_\Omega u - a \int_\Omega \frac{u^2}{v} - \frac{b}{2}c_0 \int_\Omega \frac{|\nabla v|^4}{v^3} - \frac{b}{4} \int_\Omega \frac{|\nabla v|^2}{v}.$$

In order to finally still have some control over $\int \frac{|\nabla u|^2}{u}$, as required for Lemma 2.6, we retain a small portion of this term when applying Young's inequality:

$$- \int_\Omega \frac{|\nabla u|^2}{u} + \left(\chi + 2a + \frac{b}{2}\right) \int_\Omega \frac{\nabla u \cdot \nabla v}{v} \leq \big(-1 + (1-\delta)\big) \int_\Omega \frac{|\nabla u|^2}{u} + \frac{(\chi + 2a + \frac{b}{2})^2}{4(1-\delta)} \int_\Omega \frac{u|\nabla v|^2}{v^2},$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{a,b}(u,v)(t) \leq - \delta \int_\Omega \frac{|\nabla u|^2}{u} + \left(\frac{(\chi + 2a + \frac{b}{2})^2}{4(1-\delta)} - a\chi - a - \frac{b}{4} \right) \int_\Omega \frac{u|\nabla v|^2}{v^2}$$
$$+ a \int_\Omega u - a \int_\Omega \frac{u^2}{v} - \frac{b}{2}c_0 \int_\Omega \frac{|\nabla v|^4}{v^3} - \frac{b}{4} \int_\Omega \frac{|\nabla v|^2}{v}.$$

By virtue of the presence of $-\int_\Omega \frac{|\nabla v|^4}{v^3}$, which originates from the additional summand of the energy functional and the preparations of Section 3, we can continue estimating $\int_\Omega \frac{u|\nabla v|^2}{v^2}$ by $\int_\Omega \frac{u^2}{v}$ and $\int_\Omega \frac{|\nabla v|^4}{v^3}$ and still hope for negative coefficients in front of the integrals, in contrast to the situation of Lemma 4.5. In doing so we keep some part of $\int_\Omega \frac{u^2}{v}$ for the sake of a later application of Lemma 4.4 and arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{a,b}(u,v)(t) \leq - \delta \int_\Omega \frac{|\nabla u|^2}{u} + a(1-\delta) \int_\Omega \frac{u^2}{v} + \frac{1}{4a(1-\delta)} \left(\frac{(\chi + 2a + \frac{b}{2})^2}{4(1-\delta)} - a\chi - a - \frac{b}{4} \right)_+^2 \int_\Omega \frac{|\nabla v|^4}{v^3}$$
$$+ a \int_\Omega u - a \int_\Omega \frac{u^2}{v} - \frac{b}{2}c_0 \int_\Omega \frac{|\nabla v|^4}{v^3} - \frac{b}{4} \int_\Omega \frac{|\nabla v|^2}{v},$$

which amounts to (13). ☐

Lemma 4.8. *Let $a > 0$, $b \geq 0$, $\chi > 0$ be such that*

$$\varphi(a,b;\chi) := \left(\frac{1}{4a} \left(\frac{(\chi + 2a + \frac{b}{2})^2}{4} - a\chi - a - \frac{b}{4} \right)_+^2 - \frac{bc_0}{2} \right) < 0, \tag{14}$$

and let (u,v) be a solution of (1)*. Then there are $\kappa, \delta > 0$ and $c > 0$ such that for any $t \in (0, T_{max})$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{a,b}(u,v)(t) + \kappa \mathcal{F}_{a,b}(u,v)(t) + \delta \int_\Omega \frac{|\nabla u(t)|^2}{u(t)} \leq c.$$

Proof. By continuity of

$$\delta \mapsto \varphi_\delta(a,b;\chi) := \left(\frac{1}{4a(1-\delta)}\left(\frac{(\chi+2a+\frac{b}{2})^2}{4(1-\delta)} - a\chi - a - \frac{b}{4}\right)_+^2 - \frac{bc_0}{2}\right)$$

in $\delta = 0$, for fixed a, b, χ, negativity of $\varphi(a,b;\chi)$ entails the existence of $\delta > 0$ so that $\varphi_\delta(a,b;\chi)$ is negative as well. Therefore, by Lemma 4.7,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{a,b}(u,v) + \delta\int_\Omega \frac{|\nabla u|^2}{u} + \delta\int_\Omega \frac{u^2}{v} + b\int_\Omega |\nabla\sqrt{v}|^2 \le a\int_\Omega u \qquad \text{on } (0,T_{max}).$$

Since $\int_\Omega u$ is constant in time by (8), Lemma 4.4 implies the assertion. □

Lemma 4.9. *If*

$$\chi_0 \in \left\{\chi > 0; \textit{there are } a > 0 \textit{ and } b \ge 0 \textit{ such that } \varphi(a,b;\chi) < 0\right\} =: M,$$

then

$$(0,\chi_0) \subset M.$$

Proof. Since, for any fixed $a > 0$, $b \ge 0$,

$$\chi \mapsto \varphi(a,b;\chi) = \frac{1}{64a}\left(\left(\chi^2 + 4a^2 + \frac{b^2}{4} + b\chi + 2ab - 4a - b\right)_+^2 - 32abc_0\right)$$

is monotone, for any $a > 0$, $b \ge 0$

$$\varphi(a,b;\chi_0) < 0 \text{ implies } \varphi(a,b;\chi) < 0 \text{ for any } 0 < \chi < \chi_0.$$ □

Lemma 4.10. *There is $\chi_0 > 1$ such that $\varphi(a,b;\chi_0) < 0$ for some $a > 0, b > 0$.*

Proof. Since $\varphi(\frac{1}{2},0,1) = 0$ and

$$\frac{d}{db}\varphi\left(\frac{1}{2},b,1\right)\Bigg|_{b=0} = \frac{d}{db}\left(\frac{1}{32}\left(\frac{b^2}{4}+b\right)^2 - \frac{1}{2}c_0 b\right)\Bigg|_{b=0} = \left[\frac{1}{16}(\frac{b^2}{4}+b)(\frac{b}{2}+1) - \frac{1}{2}c_0\right]\Bigg|_{b=0} = -\frac{c_0}{2} < 0,$$

there is $b > 0$ such that $\varphi(\frac{1}{2},b,1) < 0$ and by continuity of φ with respect to χ, the assertion follows. □

Proof of Theorem 1.1. By Lemma 4.10, there are $a, b > 0$, $\chi_0 > 1$ such that $\varphi(a,b,\chi_0) < 0$ and hence, by Lemma 4.9, also $\varphi(a,b,\chi) < 0$ for $\chi \in (0,\chi_0)$. An application of Lemma 4.8 thus reveals that for all $t > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{a,b}(u,v)(t) + \kappa\mathcal{F}_{a,b}(u,v)(t) + \delta\int_\Omega \frac{|\nabla u|^2}{u} \le c \tag{15}$$

for some $\kappa, \delta, c > 0$. Together with the boundedness of $\mathcal{F}_{a,b}(u,v)$ from below by Lemma 4.3 i) this ensures that $\mathcal{F}_{a,b}(u,v)$ is bounded so that an integration of (15) also shows the boundedness of $\int_t^{t+1}\int_\Omega \frac{|\nabla u|^2}{u}$. Since $\mathcal{F}_{a,b}(u,v)$ is bounded, by Lemma 4.3 ii) the same holds true for $\int_\Omega u\ln u$ and so the conditions of Lemma 2.6 are met and Theorem 1.1 follows. □

Remark 4.11. *Assuming $c_0 = \frac{1}{(2+\sqrt{2})^2}$, as permitted by Remark 3.6,*

$$-1.1 \cdot 10^{-5} \approx \varphi(0.49, 0.001; 1.015) < 0,$$

i.e. $\chi_0 > 1.015$.

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